The impact of monetary policy on bond prices: a new statistical approach.

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Abstract

Dynamical term structure models are traditionally used in finance as statistical tools for bond portfolio hedging and immunization and, more recently, to modelize the stochastic response of the term structure to shocks on the economic factors.

Here we analyze the Balduzzi, Das, Foresi and Sundaram (1996) term structure model and show that the Impulse Response function and the duration measure are closely related. Moreover, we quantify the market value of an anticipation on the monetary authority decisions. This value is uniquely determined by the no arbitrage hypothesis and by the optimality of agent trading decisions with respect to the anticipation.

Keywords: Affine Terms Structure Models, Weak Information, Impulse Response function, Monetary Policy, Stochastic Jacobian.

1 Introduction

According to Fama (1970), market informational efficiency is determined by the ability to quickly transfer information into prices. Traditional continuous

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time financial market models focus, on the contrary, on the relative consistency of prices as implied by no arbitrage restrictions. A notable exception on this line of thought is the model of Piazzesi (2003), where bond markets interact with the monetary authority decisions.

In the present paper we focus on the Balduzzi, Das, Foresi and Sundaram (BDFS 1996) model and discuss within this model the application of new statistical tools to describe and quantify market reactions to monetary authority decisions and news arrivals. These techniques extend traditional financial econometric tools and integrate them with standard continuous time stochastic models of bond markets. We show that a careful use of stochastic analysis does provide a deeper insight and a cleaner relation between macroeconomic expectational models and arbitrage based yield curve dynamics. The paper is organized as follows: in section 2, we introduce the market model and solve the pricing problem; in section 3 we discuss the Impulse Response function (IRf) in continuous time. In particular we show that the affine property of the term structure model implies an important and testable relation between the IRf and the duration measure as defined in classical immunization theory. Finally, in section 4, we introduce a new technique to evaluate the price impact of privileged and anticipative information on the monetary authority announcements.

2 The BDFS model.

The BDFS model belongs to the class of Affine Term Structure models of Duffie and Kan (1996) and Dai and Singleton (2000) where the drift vector \( \mu (X_t) \), the covariance matrix \( \sigma (X_t) \sigma^T (X_t) \) and the short rate \( R_t (X_t) \) are affine functions of the vector of factors \( X_t \). The factors are the short rate itself \( r_t \), its central tendency \( \theta_t \) and its volatility \( v_t \), so that \( X_t = (v_t, \theta_t, r_t)' \) and \( R_t (X_t) = r_t \). While \( r_t \) and \( \theta_t \) have gaussian conditional distributions, the positivity of the volatility is granted assuming a square root process for \( v_t \). Under the historical measure the dynamics of the factors will be:

\[
\begin{align*}
    dv_t &= \mu (\bar{v} - v_t) dt + \eta \sqrt{v_t} dW^v_t \\
    d\theta_t &= \alpha (\bar{\theta} - \theta_t) dt + \zeta dW^\theta_t \\
    dr_t &= \kappa (\theta_t - r_t) dt + \sqrt{v_t} dW^r_t
\end{align*}
\] (1)

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where $W_t = (W^p_t, W^v_t, W^r_t)^T$ are standard independent brownian motions under the historical measure $\mathbb{P}$ and the market information flow is described by a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by the brownian motions and completed with $\mathbb{P}$ null sets.

According to the traditional point of view, this model consists of a stochastic volatility factor model for the short rate with the addition of a gaussian stochastic factor for the central tendency of the short rate; this last factor takes into account of the additional uncertainty on the movements of the curve induced by the monetary authority decisions, which essentially control the very short end of the term structure, in particular in the case of the US market, it conveys the impact of the Federal Open Market Committee (FOMC) decisions on the bond yields as discussed in BDFS (1996).

As usual in continuous time models of bond markets, we assume that there exists a continuous set of bonds, one for each time to maturity $\tau \geq 0$ with price $P_t(\tau)$ at time $t$. In addition there exists a state price deflator $\pi(t)$ such that deflated bond prices follow a martingale under $\mathbb{P}$:

$$P_t(\tau) = E^p_t \left[ \frac{\pi(s)}{\pi(t)} P_s(\tau) \right]$$

the dynamics of $\pi(t)$ is fixed by:

$$\frac{d\pi(t)}{\pi(t)} = -r_t dt - \lambda_v \eta \sqrt{\nu_t} dW_v^r - \lambda_\theta \zeta dW_\theta^r$$

where:

$$(0, \lambda_\theta \zeta, \lambda_v \eta \sqrt{\nu_t})$$

denotes the vector of risk premia which relate the hysterical probabilities with the risk neutral ones. We prefer to keep the restrictive assumption, $\lambda_r = 0$ for sake of clarity, since it is the only way to keep unchanged the dynamics of the short rate while shifting to the pricing measure. We refer to Grasselli and Tebaldi (2004) where the same results are formally discussed in a general affine term structure model.

A zero coupon bond expiring at $t + \tau$ obeys the boundary condition: $P_{t+\tau}(0) = 1$. Hence using the Girsanov theorem, we can write the bond price as follows:

$$P_t(\tau) = E^p_t \left[ \frac{\pi(t + \tau)}{\pi(t)} \right] = E^Q_t \left[ \exp \left( - \int_t^{t+\tau} r_s ds \right) \right]$$

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where \( Q \) denotes the risk neutral measure defined by:

\[
\frac{dQ}{dP} \bigg|_T = \pi(T) \exp \left( \int_0^T r_s \, ds \right)
\]

The effect of the change of measure in this case is simply the change in the drifts of the factors’ SDE:

\[
\alpha (\bar{\theta} - \theta) \rightarrow \alpha (\tilde{\theta} - \theta)
\]

\[
\mu (\bar{\nu} - \nu_t) \rightarrow \tilde{\mu} (\tilde{\nu} - \nu_t)
\]

The model belongs to the class of canonical models as defined in Dai and Singleton (2000), in fact the discounted characteristic function of the vector of factors \( X_t \), conditional on the information at time \( t \leq T \)

\[
\Psi_X (u, X_t, t, T) = \mathbb{E}^Q_t \left[ \exp \left( - \int_t^T r(s) \, ds \right) \exp (u' X_T) \right], \quad u \in \mathbb{C}^3. \tag{3}
\]

can be explicitly written as an exponential affine function of the factors:

\[
\Psi_X (u, X_t, t, T) = \exp (\mathcal{V}^0 (T - t) + \mathcal{V} (T - t)' X_t), \quad t \leq T,
\]

where \( \mathcal{V} \in \mathbb{C}^3 \) satisfy the following complex-valued backward (Riccati) ODE Riccati equations \( (\tau = T - t) \):

\[
\frac{d}{d\tau} \mathcal{V}_v(\tau) = -\mu \mathcal{V}_v(\tau) + \frac{1}{2} (\eta \mathcal{V}_v^2(\tau) + \mathcal{V}_r^2(\tau)), \quad \mathcal{V}(0) = u \in \mathbb{C}^3
\]

\[
\frac{d}{d\tau} \mathcal{V}_\theta(\tau) = -\alpha \mathcal{V}_\theta(\tau) + \kappa \mathcal{V}_r(\tau)
\]

\[
\frac{d}{d\tau} \mathcal{V}_r(\tau) = -\kappa \mathcal{V}_r(\tau) - 1
\]

which have the closed form solution \( (\alpha \neq \kappa) \):

\[
\mathcal{V}_v(\tau) = u_r e^{-\kappa \tau} - \frac{1 - e^{-\kappa \tau}}{\kappa}
\]

\[
\mathcal{V}_\theta(\tau) = u_\theta e^{-\alpha \tau} - \frac{1 - e^{-\alpha \tau}}{\alpha} + (\kappa u_r + 1) \frac{e^{-\kappa \tau} - e^{-\alpha \tau}}{\alpha - \kappa}
\]

while (see Grasselli and Tebaldi 2004b):

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\[ V_v(\tau) = \frac{b_1(\tau, u_r) u_v + b_2(\tau, u_r)}{b_3(\tau, u_r) u_v + b_4(\tau, u_r)} \]

with:
\[
\begin{pmatrix}
  b_1(\tau, u_r) & b_2(\tau, u_r) \\
  b_3(\tau, u_r) & b_4(\tau, u_r)
\end{pmatrix} = \exp\left( -\frac{\mu \tau}{2} + \frac{1}{2} \int_0^\tau \gamma^2_v(s) \, ds \right)
\]

which can be computed explicitly obtaining:
\[
\begin{pmatrix}
  \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{\lambda_1} - e^{\lambda_2}) \\
  \frac{2 \lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{\lambda_1} - e^{\lambda_2})
\end{pmatrix} = \frac{\mu}{\eta \tau} \begin{pmatrix}
  e^{\lambda_1} - e^{\lambda_2} \\
  e^{\lambda_2} - e^{\lambda_1}
\end{pmatrix}
\]

with:
\[
\lambda_{1,2} = \frac{-\mu \tau \pm \sqrt{(\mu \tau)^2 - 4 \eta \tau \int_0^\tau \gamma^2_v(s) \, ds}}{2}
\]
\[
\int_0^\tau \gamma^2_v(s) \, ds = \left( u_r + \frac{1}{\kappa} \right) \frac{2 - e^{-2\kappa \tau}}{2\kappa} - \frac{1}{\kappa} \left( u_r + \frac{1}{\kappa} \right) \frac{1 - e^{-\kappa \tau}}{\kappa} + \frac{\tau}{\kappa^2}
\]

**Remark 1** The bond price (boundary condition \( V(0) = 0 \)) is given by:
\[
\begin{align*}
V_r(\tau) &= -\frac{1 - e^{-\kappa \tau}}{\kappa} \\
V_\theta(\tau) &= -\frac{1 - e^{-\alpha \tau}}{\alpha} + \frac{e^{-\kappa \tau} - e^{-\alpha \tau}}{\alpha - \kappa} \\
V_v(\tau) &= -\frac{1}{2} \left( \int_0^\tau \gamma^2_v(s) \, ds \right) \frac{1 - e^{2\sqrt{\Delta}}}{\lambda_1 - \lambda_2 e^{2\sqrt{\Delta}}} \\
\Delta &= (\mu \tau)^2 - \eta \tau \left( \int_0^\tau \gamma^2_v(s) \, ds \right)
\end{align*}
\]

As observed in Dai Singleton (2000), the BDFS is very restrictive in the description of the correlation structure between the volatility, the central tendency and the short rate. These restrictions allow a direct identification of the factors with (almost) observable and economically significant quantities like the central tendency of the short rate \( \theta \) and its volatility \( v \).
3 Impulse-Response function.

The statistical analysis of the market reaction to monetary policy is a traditional and largely investigated topic in financial econometrics, see e.g. Hamilton and Jordà (2002) for a survey on recent literature. An important contribution that takes into account the restrictions imposed by the No Arbitrage principle on price evolution can be found in Piazzesi (2003). In that paper, an affine continuous time model (not far from the BDFS one) has been introduced; however, as observed in Hamilton and Jordà (2002), the statistical inference of the results in that context are much more demanding than traditional discrete-time models.

In order to gain a deeper understanding of the financial implications in continuous time, we are going to discuss the definition of the Impulse Response function, which is an essential tool to investigate the relation between prices and factors.

The Impulse Response function is traditionally defined for a VAR model as a time dependent matrix $M_{t,t+k} = (m_{t,t+k})_{i,j=1,...,n}$, describing the response at time $t + k$ of factor $i$ to a shock at time $t$ on the factor $j$ innovation.

A formal discussion and derivation of such quantity in continuous time has been given in Grasselli and Tebaldi (2004), where they observe that the infinitesimal counterpart of the IRf is simply given by the Malliavin derivative of the factors’ process:

$$D_t X_T = \hat{D}_{t,T} \sigma \left( X_t \right) 1_{t \leq T}.$$ 

The first order variation process $\hat{D}_{t,T}$ is obtained by a formal differentiation of the factor processes:

$$\left( d\hat{D}_{s,t} \hat{D}^{-1}_{s,t} \right)_{i,j} = \frac{\partial \mu \left( X_t \right)_i}{\partial X_{t,j}} dt + \frac{\partial \left( \sigma \left( X_t \right) dW_t \right)_i}{\partial X_{t,j}} + \int_{s}^{t} \left( \begin{array}{cc}
- \mu & 0 \\
0 & -\alpha & 0 \\
0 & \kappa & -\kappa
\end{array} \right) dt + \int_{s}^{t} \left( \begin{array}{cc}
\eta \frac{dW_{\nu}}{\sqrt{\nu}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array} \right) dW_{\nu}$$

$$\hat{D}_{s,s} = I$$

In the familiar case of a Vector AutoRegressive (VAR) model, the IRf is deterministic, while in a generic affine model is stochastic: therefore its
expression depends on the path followed by the process. An empirical estimation of such quantity requires a critical revision of the IRf definition in continuous time. First of all, it seems more appropriate for an econometric identification to consider the variance-scaled process:

\[ \hat{D}_{t,T} = D_t X_T \sigma^{-1}(X_t) 1_{t \leq T}. \]

In fact, a separate estimation of the volatility term gets rid of the stochasticity induced by the state dependent factor volatilities. When dealing with the term \( \hat{D}_{t,T} \), a possible separation between the trend and the noise components is naturally suggested by introducing the \( T \)-forward measure, i.e. by choosing the bond expiring at time \( T \) as numeraire. The corresponding change of measure is given by

\[ \frac{dQ^T}{dQ} |_{s,t} = \exp\left( -\int_s^t \gamma_l dt \right) \frac{P_l(T-t)}{P_s(T-s)}, \]

Although quite unusual in econometric approaches, the choice of the bond price as the numeraire has a crucial advantage: as shown in Elliott and Van Der Hoek (2001), the \( T \)-forward conditional expectation \( D^T_{s,t} \) is a deterministic function and can be computed in terms of the solution of the Riccati ODE, as discussed in Grasselli and Tebaldi (2004). The remaining stochastic term is a white noise correction:

\[ \hat{D}^T_{s,t} = D^T_{s,t} \exp \left( N_{s,t} - \frac{1}{2} < N, N >_{s,t} \right), \]

where

\[ D^T_{s,t} = E^T_t \left[ \hat{D}^T_{s,t} \right] = \exp \left( \begin{array}{ccc} -\mu (t-s) + \frac{1}{2} \int_s^t \eta^2 v_r (\tau) d\tau & 0 & 0 \\ 0 & -\alpha (t-s) & 0 \\ +\frac{1}{2} \int_s^t \gamma_r (\tau) d\tau & \kappa (t-s) & -\kappa (t-s) \end{array} \right) \]

\[ N_{s,t} = \frac{1}{2} \left( \begin{array}{ccc} \int_s^t \eta (\sqrt{v_r})^{-1} dW^T_{v,\tau} & 0 & 0 \\ 0 & 0 & 0 \\ \int_s^t (\sqrt{v_r})^{-1} dW^T_{r,\tau} & 0 & 0 \end{array} \right). \]

It is important to notice that \( \hat{D}^T_{s,t} \) is the natural multifactor generalization of the well known duration measure used in traditional bond immunization.
theory. This extends to the BDFS model the traditional argument used in CIR (1985) and in Brown and Schaefer (1994): the sensitivity of the yield curve with respect to unexpected shocks on the short rate is easily obtained in terms of the relation:

$$\frac{\partial P_t(\tau)}{\partial r_t} = \mathcal{V}_r(t) P_t(\tau)$$

where $\mathcal{V}_r(t)$ plays the same role of (minus) the duration measure in classical deterministic immunization. Elliott and Van Der Hoek (2001) showed that in a generic affine term structure model the following relation holds:

$$\mathcal{V}(t) = -\int_t^T ds \left( D^T_{t,s} \right)' \gamma$$

$$R_t(X_t) = \gamma_0 + \gamma' X_t.$$ 

If we apply to the BDFS model, with $\gamma = (0, 0, 1)'$, this relation becomes:

$$\mathcal{V}_r(t) = -\int_t^T ds \left( D^T_{t,s} \right)_{3,3}'$$

We conclude this section by providing the explicit computation of the matrix $D^T_{s,t}$ in the BDFS model:

$$D^T_{s,t} = \begin{pmatrix}
ed_1 & 0 & 0 \\
0 & e^{-\alpha(t-s)} & 0 \\
\delta_{3,1} & \frac{\kappa}{\kappa-\alpha} (e^{-\alpha(t-s)} - e^{-\kappa(t-s)}) & e^{-\kappa(t-s)}
\end{pmatrix}$$

$$d_1 = -\mu(t-s) + \frac{1}{2} \int_s^t \eta^2 \mathcal{V}_v(\tau) d\tau,$$

$$\delta_{3,1} = \frac{\left( -\int_s^t \mathcal{V}_r(\tau) d\tau \right) (e^{d_1} - e^{-\kappa(t-s)})}{2(\kappa-\mu)(t-s) + \int_s^t \eta^2 \mathcal{V}_v(\tau) d\tau}.$$

The structure of these equations deserves some comments. In particular, the triangular form of $D^T_{s,t}$ implied by the factors’ definition poses severe restrictions on the response of the short rate to an innovation in the monetary policy. In fact, the conditions on the drift, as discussed in the previous section, preclude cross correlations between the monetary policy and volatility factors, which should be important to take into account (e.g. ”volatility effect” of the monetary policy).
4 The value of an anticipation on the Fed decisions

Beyond the above reformulation of traditional statistical tools used in financial econometrics, the final contribution of this paper is the introduction of a new technique to quantify the market value of a privileged information. This formulation identifies uniquely the expected price variation, conditional to a specified information set.

Consider a small investor (price taker) possessing some personal information about the future decision of the central bank. Obviously, he will trade trying to exploit the value of his personal knowledge. We pursue further the ”weak information” approach introduced in Baudoin (2003): the information set is modelled by a stochastic function $Y_T$ of the factors, while the agent’s beliefs are encoded by a probability measure $\nu$ which represents the ”personal prediction” of the law for $Y_T$. Rather than discussing the technical requirements which have to fulfill $\nu$ and $Y_T$ (see Baudoin 2003), we focus on the economic financial implications of this model.

The change in the a priori beliefs implies different expected risk reward profiles of the agents. From the probabilistic point of view, this can be interpreted as a shift from the reference measure $\mathbb{P}$ of the representative agent (who reflects the common beliefs of the market), to $\mathbb{P}_\nu$, which represents the a priori beliefs of the informed agent. The corresponding extra expected risk premium can be identified with the value conveyed by such privileged information.

Baudoin (2002) showed that there exists a unique pricing measure $\mathbb{P}_\nu$ which is consistent with an agent that trades trying to fully exploit the advantage coming from the knowledge of the law $\nu$ for a measurable functional $Y$ of the factors.

Let us now apply the weak information approach in our concrete example. We consider the BDFS model and we quantify the value of an anticipation on a decision of the monetary authority. Consider a meeting of the Federal Open Maket Committee (FOMC) which is planned at time $T_{FED}$; the FOMC will take under consideration only two possibilities:

- to keep the level of the fed fund target fixed at level $\bar{\theta}$ or
- to increase the target to a new level $\bar{\theta}^N$.

Suppose that a financial institution has developed an internal model in order to anticipate the choices of the authority. The forecasting model for-
ulates a prediction about the decision on the basis of the factors’ level and predicts that the conservative choice will occur with a probability $p$.

A rule of this type has been termed a "high frequency policy rule" in the terminology of Piazzesi (2003): it describes, for a given value of the factors, the predictable component of the central bank decision. Observe that the beliefs of the informed agent are represented by the probability $p$ which eventually can be different from those shared by the market (representative agent).

According to the above discussion, the change in the belief is reflected in a change of the reference measure of the agent. As a consequence, the optimal trading strategy followed by a utility maximizing investor will change. In fact, the passage to the risk neutral pricing measure will imply different expected risk premia and as a consequence a modified trading strategy. While we refer to the original reference Baudoin (2002) for the utility maximization procedure, we focus here on the proper change of measure which describes, in the BDFS model, the anticipation about the target change, which is traditionally identified by the central tendency parameter of the monetary policy factor $\bar{\theta}$.

It is then immediate to determine the change of measure reflecting the change in the beliefs as:

$$
\frac{d\mathbb{P}_\nu}{d\mathbb{P}} \bigg|_T = \frac{p}{\Pr_{Mkt} (NoCh)} \mathbb{I}_{NoCh}(\omega) + \frac{1 - p}{1 - \Pr_{Mkt} (NoCh)} (1 - \mathbb{I}_{NoCh}(\omega))
\cdot \exp \left( - \int_s^T \frac{\zeta (\bar{\theta}^N - \bar{\theta})}{\zeta} \mathbb{I}_{t>T_{FED}} dW_t^g - \frac{\alpha^2 (\bar{\theta}^N - \bar{\theta})^2}{2\zeta^2} \mathbb{I}_{t>T_{FED}} dt \right)
$$

where $\Pr_{Mkt} (NoCh)$ corresponds to the probability that the central bank does not modify the target according to the market (representative agent) expectations.

The indicator function $\mathbb{I}_{NoCh}(\omega)$ selects the trajectories for which, according to the policy rule, a change in the target should not occur.

When the policy rule predicts a change of the target (associated to the indicator $\mathbb{I}_{Ch}(\omega) = 1 - \mathbb{I}_{NoCh}(\omega)$), the central tendency $\bar{\theta}$ shifts to a new value $\bar{\theta}^N$ starting from $T_{FED}$. 
The private information in this case is represented by the probability $p$ which can be different from $\Pr_{Mkt}(NoCh)$. The term "weak" is related to the fact that all agents share the policy rule (mathematically $p \neq 0$ iff $\Pr_{Mkt}(NoCh) \neq 0$).

As usual the Girsanov change of measure implies that the evolution of the prices under the new measure after the announcement will differ from those under the hystorical measure for a drift term. It will quantify the bias in the expectations due to the private beliefs of the financial institution, i.e. the value of the privileged information.

In order to compute such drift the indicator function has to be expressed as a function of the factors: $\mathbb{I}_{NoCh}(\omega) = f(t, \tau_t(\omega), \theta_t(\omega), \nu(\omega))$.

Substituting the expression for $\mathbb{I}_{NoCh}(\omega)$ in terms of $f$, the change of measure becomes a function of the factors:

$$\left[ \frac{dP_{\nu}}{dP} \right]_t \triangleq \xi(X_t).$$

A martingale representation theorem implies (see Baudoin 2003) that the drift has a general expression in terms of the policy rule and of the IRf:

$$ID_t = D_t \log E^p_t [\xi(X_T)]$$

$$= E^p_t [\nabla \xi(X_t) D_t X_T].$$

The final computation involves the expression of the IRf, which has been discussed in the previous section, and the gradient of the change of measure that depends on the policy rule. The policy rule proposed in Piazzesi (2003) is linear in the factors, thus the computation simplifies and $ID_t$ becomes a linear function of $E^p_t [D_t X_T]$.

It is natural to conclude this section by mentioning a largely known and documented empirical evidence for a bias in the beliefs of the market which has been discussed for example in Balduzzi, Bertola, Foresi and Klapper (1996). These authors observed that empirical data on the term fund rates displayed volatile and persistent spreads from the target. This behavior is perfectly consistent with the approach we discussed above. In fact, our model formalizes an expectational model of short term rates where a specific belief of the investor induces a bias on the expectations about observable rates.
References


