Discrete and continuous time approximations of the optimal exercise boundary of American options*

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Abstract

The valuation of American-style options gives rise to an optimal stopping problem involving the computation of a time dependent exercise boundary over the whole life of the contract. An exact computational formula for this time dependent optimal boundary is not known. Nevertheless, some numerical approaches can be proposed to approximate the optimal boundary.

In particular, in this contribution we study three different numerical techniques: an improved lattice method, a randomization approach based on the American option valuation procedure proposed by Carr and an analytic approximation procedure proposed by Bunch and Johnson.

The three techniques studied are tested and compared through a wide empirical analysis.

Keywords. Option pricing, American options, optimal exercise boundary

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1 Introduction

The early exercise of an American option depends on the comparison between the current price of the underlying security and a critical value. In particular, for American put options it is

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optimal to exercise in advance if the current price is sufficiently low, while for American calls early exercise is convenient if the current price is sufficiently high. As time to maturity varies, this critical price changes, too.

The critical prices all together form a function of time which is called optimal exercise boundary. Hence, a correct understanding of the behaviour of the optimal exercise boundary is crucial for valuing American-style options.

The valuation of American-style securities gives rise to an optimal stopping problem involving the computation of the time dependent exercise boundary over the whole life of the option. Unfortunately, an analytic formula for this time dependent optimal boundary is not known.

The main purpose of this contribution is to study some procedures which enable us to approximate the optimal exercise boundary.

Actually, if we are interested only in the computation of the option price, it is not essential to use a highly precise approximation of the optimal boundary (see for example [Ju, 1998]). Nevertheless, the computation of the optimal boundary is crucial if we aim at evaluating other important information, such as the average exercise time, the probability of early exercise, the overall probability that the option is exercised, and so on. To this aim, if we were able to determine a good approximation of the optimal boundary, it would be possible to use it in a Monte Carlo framework.

Different numerical approaches can be proposed to approximate the optimal boundary. In particular, in this contribution we study three different numerical techniques:

1. a lattice approach based on the well-known binomial CRR model ([Cox, Ross and Rubinstein, 1979]);

2. a randomization approach based on an American option valuation procedure proposed by [Carr, 1998];

3. an analytic approximation approach proposed by [Bunch and Johnson, 2000].

In the last two decades, a number of contributions have appeared in the literature which try to use in some way the optimal exercise boundary in order to compute the fair value of an American option. Among these, we found [Brenner, Courtadon and Subrahmanyam, 1985], [Omberg, 1987], [Kim, 1990], [Jacka, 1991], [Carr, Jarrow and Myneni, 1992], [Myneni, 1992], [Ju, 1998], [Longstaff and Schwartz, 2001], [Little, Pant and Hou, 2000]. However, the contributions of [Carr, 1998] and [Bunch and Johnson, 2000] seem the most adapt to be used in order to obtain a sufficiently accurate approximation of the optimal exercise boundary of American options.

In the following, the analysis is carried out for American put options, but can be extended to American calls by exploiting some useful symmetry results. In particular, there exist both a parity relation which connects the prices of American call and put options and a symmetry property which relates the relevant critical prices (see [Detemple, 2001] for a comprehensive discussion).

As regards the procedures analyzed, the lattice procedure proposed tries to improve the fluctuating behaviour of the optimal exercise boundary in a binomial model by looking for a smoother boundary which is obtained by means of an interpolation technique.

Both Carr’s and Bunch and Johnson’s procedures allow to determine an approximation of the critical price at current time, but can be used iteratively in order to obtain the approximated value of the boundary in the subsequent times until maturity. More precisely, the exercise boundary
can be “built” by computing the critical prices in correspondence with a sufficiently dense grid of time values.

The three techniques studied are tested and compared through a wide empirical analysis carried out by considering a number of different option valuation problems generated through a Monte Carlo simulation procedure.

The paper is organized as follows. Section 2 briefly presents the optimal exercise boundary and discusses its main properties in continuous time. Section 3 analyzes the optimal exercise boundary in a lattice approach while Section 4 proposes the interpolated boundary in the context of the Cox, Ross and Rubinstein binomial model. Section 5 presents Carr’s randomization procedure and Section 6 discusses Bunch and Johnson’s approximations. Section 7 presents the empirical experiments carried out and, finally, Section 8 suggests some concluding remark.

2 The early exercise boundary

We assume that the price $S_t$ of the underlying asset is governed by a risk neutralized diffusion process

$$dS_t = S_t[r dt + \sigma dW_t],$$

where $W_t$ is a standard Wiener process, $r > 0$ is the risk-free interest rate and $\sigma$ is the volatility of the asset returns; both parameters are supposed constant.

Moreover, let $t = 0$ be the current time, $t = T$ the option maturity and let us consider an American style put option.

It is known that, according to the arbitrage pricing theory, the fair value at time $t = 0$ of a put option can be obtained by solving the following optimal stopping problem

$$P_0 = \sup_{t \in [0, T]} \mathbb{E}\left[e^{-rt}(X - S_t)^+\right],$$

where the expected value $\mathbb{E}$ is computed using the risk neutral probability measure with respect to the information set of time $t = 0$ (see e.g. [Karatzas and Shreve, 1998], [Shiryaev et al., 1994]). Since the holder has the right to exercise the option at any time, the supremum is taken over the class of all possible exercise times.

Let us define the function

$$B : [0, T] \rightarrow \mathbb{R}^+$$

which gives for each time $t \in [0, T]$ the critical exercise prices below which it is optimal to exercise the American put. The function $B$ is known as the early exercise boundary or optimal exercise boundary.

Formally, the early exercise boundary can be defined as the optimal solution of the following problem of first passage time through a boundary ([Carr, 1998], [Bunch and Johnson, 2000])

$$P_0 = \sup_{B} \mathbb{E}\left[e^{-rt_B}(X - S_{t_B})^+\right]$$

where $S_0$ is assumed greater than $B_0$ and the stopping time $t_B$ is given by the first passage time of $S$ through the exercise boundary $B$

$$t_B = \inf \{\{ t \in [0, T] : S_t \leq B_t \} \cup \{ T \}\}.$$
Note that the stopping time $t_B$ will in general be different along different trajectories of the price process $S_t$.

From a financial point of view, the early exercise boundary is a path of critical underlying prices at which it is optimal to exercise the option. Indeed, the value at time $t$ of the early exercise boundary, $B_t$, represents the critical price of the underlying asset with which the holder of the American option has to compare the current asset price $S_t$ in order to decide whether it is optimal to immediately exercise the option or to continue with it.

The early exercise boundary divides the time-asset price space $\{(t, S)\}$ into two regions: the continuation region $C$ and the stopping region $S$. For a put option, the continuation and the stopping regions are defined as follows

$$C = [0, T] \times (B_t, +\infty)$$
$$S = [0, T] \times [0, B_t].$$

Therefore, if the early exercise boundary were known, we would have an optimal stopping rule to decide the exercise strategy. Unfortunately, the function $B_t$, which represents the boundary that separates $C$ and $S$, is not known a priori, but must be determined as part of the solution to the option valuation problem.

The critical price of the underlying asset which separates the exercise region of prices from the continuation region at a given time $t \in [0, T]$ for an American put option can be characterized in different ways. In particular, the following properties can be equivalently used to define the critical price:

1. the asset price below which it is optimal to exercise the American put;
2. the asset price at which one is indifferent between exercising and continuing with the option;
3. the highest value of the asset price for which the put value $P_t$ is equal to the exercise price $X$ less the stock price $S_t$, i.e. $P_t = X - S_t$;
4. the highest value of the underlying asset price at which the put value does not depend on time to maturity.

As regards the optimal exercise boundary $B$, the following features can be proved:

1. $B$ is continuously differentiable on the interval $[0, T]$;
2. $B$ is nondecreasing in $t$ (and therefore nonincreasing in the time to maturity $\tau = T - t$);
3. near expiration we have $B_T = \lim_{t \to T} B_t = X$;
4. $B$ does not depend on the current price of the underlying asset $S_0$;
5. $B$ is linearly homogeneous in $X$.

Properties (1)-(3) are discussed in [van Moerbeke, 1976], [Myneni, 1992], [Kim, 1990] and [Jacka, 1991]; properties (4) and (5) are analyzed in [Basso, Nardon and Pianca, 2001].

It can be proved that some symmetry properties between American calls and puts allow to extend the results obtained for the early exercise boundary of American puts also to the boundary of American calls; for a comprehensive treatment of the symmetry properties for American options see [Detemple, 2001].
3 The optimal exercise boundary in a binomial approach

It is interesting to analyze the optimal exercise boundary not only in continuous time models but also in lattice approaches. In this section we study early exercise in a classical binomial model, such as the CRR model of [Cox, Ross and Rubinstein, 1979].

Let
\[ 0, T/n, 2T/n, \ldots, T \] \hspace{1cm} (8)
be the trading dates until the option maturity \( T \) and let \( \Delta t = T/n \). By assumption, at time 0 the price of the underlying security is \( S_0 > 0 \) and the underlying security price follows a binomial random walk such that at the end of each period \( i \) the price \( S_i \), conditional to the price \( S_{i-1} \) at time \( i-1 \), can take only two values: the up price \( uS_{i-1} \) with probability \( q \) or the down price \( dS_{i-1} \) with probability \( 1 - q \).

Each node of the lattice is identified by the pair \((i, j)\) where the step \( i \), with \( 0 \leq i \leq n \), determines the time period (equal to \( i\Delta t \)) and \( j \), with \( 0 \leq j \leq i \), is the state of nature, which is completely determined by the number of upward movements in the underlying security price. Let \( P_{i,j} \) denote the price of the American put option at time \( i\Delta t \) if \( j \) upward movements occurred, i.e. if the price of the underlying security is \( S_{i,j} = S_0 u^j d^{i-j} \).

An American option can be evaluated using a dynamic programming procedure, starting at time \( T \) (stage \( i = n \)) and working backward through the binomial lattice. At expiration, the value of the American put option is
\[ P_{n,j} = (X - S_{n,j})^+ \hspace{1cm} 0 \leq j \leq n. \] \hspace{1cm} (9)

The early exercise feature of an American option entails that at each time step \( 0 \leq i < n \) the option could be exercised, if this turns out to be optimal. Therefore, both the continuation value and the exercise value have to be computed and compared in each node in order to determine the optimal strategy. Thus at each time step \( 0 \leq i < n \) the option value is given by the maximum between the continuation and the immediate exercise value
\[ P_{i,j} = \max \left[ e^{-r\Delta t} (pP_{i+1,j+1} + (1-p)P_{i+1,j}) , \ X - S_{i,j} \right] \hspace{1cm} 0 \leq j \leq i \] \hspace{1cm} (10)
where the expected continuation value in equation (18) is computed with respect to the risk neutral probability of an up movement
\[ p = \frac{e^{r\Delta t} - d}{u - d} \] \hspace{1cm} (11)
and the no arbitrage condition
\[ d < e^{r\Delta t} < u \] \hspace{1cm} (12)
is supposed satisfied. This recursive evaluation procedure can be used to obtain the present value of the American put, \( P_0 = P_{0,0} \).

We have seen that at each time step \( i < n \) early exercise might be optimal. As a matter of fact, there will exist a critical price \( B_i \), such that if \( S \leq B_i \) the put option should be exercised immediately.
The stopping and the continuation regions which partition the set of all nodes may be defined as:

\[ S = \{ (i, j) : P_{i, j} \leq X - S_t d^{j-i} \} \]  \hspace{1cm} (13)

\[ C = \{ (i, j) : P_{i, j} > X - S_t d^{j-i} \} \].  \hspace{1cm} (14)

The optimal strategy for the holder of a put option is to exercise when in \( S \), and to wait when in \( C \).

In order to characterize the optimal exercise boundary in the binomial model let us denote by \( I \) the set of time steps in which there exists at least one stopping node

\[ I = \{ i : (\exists j : (i, j) \in S) \} \].  \hspace{1cm} (15)

It is possible to show (see [Kim and Byun, 1994] and [Curran, 1995]) that, for each time step \( i \in I \), there exists a unique state \( B(i) \) such that \( (i, j) \in C \) for \( j > B(i) \) and \( (i, j) \in S \) for \( j \leq B(i) \). Thus, for each time step \( i \in I \) we can define both the optimal exercise state

\[ B(i) = \max\{ j : (i, j) \in S \} \]  \hspace{1cm} (16)

and the optimal exercise node \((i, B(i))\). Finally, the optimal exercise boundary \( B \) in this discrete setting can be defined as the set of all optimal exercise nodes

\[ B = \{ (i, B(i)) : i \in I \} \].  \hspace{1cm} (17)

The following properties show that the optimal exercise boundary \( B \) is “continuous” and “non increasing” in time to maturity.

1. The optimal exercise boundary for an American put option is continuous in the sense that either \( B(i-1) = B(i) \) or \( B(i-1) = B(i) - 1 \) for \( i, i-1 \in I \).

2. The optimal exercise boundary for an American put option is non increasing in time to maturity in the sense that if \( B(i-1) = B(i) \) then \( B(i-2) = B(i-1) - 1 \) for \( i, i-1, i-2 \in I \).

Note that the first property implies that it would not be possible for the underlying security price to be in the set of \( S \backslash B \) without crossing the optimal exercise boundary \( B \). Thus, the optimal exercise strategy is to exercise when the price \( S \) hits the optimal exercise boundary \( B \).

4 A binomial approximation to the optimal exercise boundary of continuous time options

A widely used operational device suggests to utilize a discrete lattice approach with a high number of steps in order to approximate a continuous time process. With regard to this, it is well known that the CRR binomial model converges to the diffusion process (1).

Nevertheless, at least three problems arise when a lattice procedure is used in order to approximate the optimal exercise boundary of an option written on an asset with a continuous time dynamics.
A first problem is that the lattice methods may sometimes fail to define a critical price, since in the first time steps of the lattice the existence of a stopping node is not guaranteed. Indeed, it may well happen that all the nodes in the first time steps lies above, or below, the continuous time early exercise boundary.

A second problem is due to the fact that a lattice approach can give a too coarse approximation of the optimal exercise price when there are too few nodes defined in the lattice.

Both these problems are highlighted in figure 1 which shows the behaviour of the early exercise boundary in a binomial model as the number of steps increases. Actually, from this figure we can see that the boundary is not defined in the first steps of the binomial tree; however, the amplitude of the time interval affected by this drawback decreases as the number \( n \) of steps in the tree increases. Moreover, it can be seen that when the number of steps is low, the boundary computed with the binomial tree gives a poor approximation of the continuous time exercise boundary.

A third problem is connected to the feature that the continuous time critical prices are independent of the initial stock price while it not so for the binomial model. Actually, the initial stock price \( S_0 \) does affect the construction of the tree from which the optimal exercise boundary is computed in a lattice approach.

To overcome at least partly the second and third problems, we have introduced in the com-
putation of the critical prices of the binomial model a slight modification of the boundary values $S_{i,B(i)}$, with $i \in \mathcal{I}$, above defined. More precisely, we have computed the critical price at time $t$ as an approximation of the price $S_t$ which makes the option value $P_t$ equal to the immediate exercise value $(X - S_t)^{+}$. Of course, only by chance a critical price computed in such a way will coincide with the price observed in any of the nodes defined in the lattice at stage $i = t/\Delta t$.

In particular, this approximated critical value has been computed by applying the following linear interpolation scheme between the values observed in the optimal exercise state $B(i)$ and the adjacent node $B(i) + 1$.

$$B^*(i) = w_1 S_{i,B(i)} + w_2 S_{i,B(i)+1}$$

(18)

where the weights $w_1$ and $w_2$ are determined as

$$w_1 = \frac{P_{i,B(i)+1} - (X - S_{i,B(i)+1})}{P_{i,B(i)+1} - P_{i,B(i)} + S_{i,B(i)+1} - S_{i,B(i)}}$$

(19)

$$w_2 = \frac{-P_{i,B(i)} + (X - S_{i,B(i)})}{P_{i,B(i)+1} - P_{i,B(i)} + S_{i,B(i)+1} - S_{i,B(i)}}$$

(20)

This interpolation procedure improves the precision of the approximated critical prices, and thus the accuracy of the boundary approximation, and let the boundary depend much less on the initial price $S_0$. Figure 2 shows the placement of the interpolated early exercise boundary (18) with respect to the theoretical boundary $S_{i,B(i)}$ given by the asset price observed in the boundary node $B(i)$ and the value $S_{i,B(i)+1}$ observed in the adjacent node $B(i) + 1$. It can be observed that the interpolation procedure effectively reduces the fluctuations of the critical prices.

We have tested the influence of the initial asset price on the interpolated early exercise boundaries of binomial trees through a large set of experiments carried out with different initial stock price levels. This influence affects above all the first steps. An easy way to face this problem is to make the binomial lattice start some time before the current date, say at time $t = -\bar{t}$, so that the first steps (from $-\bar{t}$ to 0) can be omitted from the computation of the boundary.

Moreover, the experiments confirm that if few time steps or extreme values for the moneyness, i.e. the ratio between the stock price and the strike price, are considered, the critical boundary may not exist at the first steps or can give a too coarse approximation of the continuous time value. On the other hand, the interpolated boundary computed with a 25 000-step binomial tree, with the omission of the first 5 000 steps, has proved very robust with respect to all of the three problems discussed.

In order to be able to use the binomial method with a high number of time steps we have coded an efficient version of this algorithm which exploits the property that the up and down step parameters of the CRR method are one the reciprocal of the other one, i.e. $d = u^{-1}$. This property entails that $S_0 u^j d^{i-j} = S_0 u^{2j-i}$ and therefore all the asset prices defined in the lattice at any stage pertain to the set

$$\{ S_0 u^j : j = -n, -n + 1, \ldots, 0, \ldots, n - 1, n \}.$$  

(21)

This allows to build all the stock prices before starting the backward iterative procedure, thus avoiding the computation of new prices at each stage. As a result, a very efficient algorithm can be coded.
Figure 2: Theoretical and interpolated optimal exercise boundaries in the CRR binomial model with $n = 52$ weekly steps and $S_0 = 100, \sigma = 0.2, r = 0.05$, for an option with strike price $X = 100$ and maturity $T = 1$. The higher curve connects the asset prices in the higher nodes adjacent to the theoretical boundary.

5 The randomization approach

[Carr, 1998] develops a new approach for determining both the value and the early exercise boundary of American options. Carr’s approach is based on a technique called randomization. According to Carr’s definition, randomization represents a procedure for computing the expected value of a random variable which entails the following three steps:

1. let the expected value of the random variable depend on a parameter which is “randomized” by assuming a plausible distribution for it;

2. calculate the expected value of the dependent variable (which is unknown in the fixed parameter model) in this random parameter setting;

3. let the variance of the distribution governing the parameter approach zero, holding the mean of the distribution constant at the fixed parameter value.

For example, if we consider a standard option, we could randomize the initial stock price, the strike price, the initial time, the maturity date and so on.

Carr uses a randomization approach in order to determine the value of an American put option and its critical stock prices; to this aim, he randomizes the maturity date of the option.
Therefore, the owner of this randomized American put can exercise at any time up to some random maturity date.

More precisely, Carr assumes that the maturity of the randomized American put is determined by a waiting time dependent on the arrivals of a standard Poisson process, which is assumed to be independent of the underlying stock price process and uncorrelated with any market factor. These assumptions permit to compute the value of the randomized American put of step 2, thus allowing the application of the randomization approach to the American option valuation problem. What is of interest with respect to the computation of the optimal exercise boundary, is that the randomization procedure gives also a formula for the boundary of the randomized option.

The randomization approach can be applied to the option valuation problem in different ways according to the probability distribution chosen for the random maturity. In particular, Carr uses either an exponential distribution or a gamma distribution with mean equal the fixed maturity $T$ of the option under examination.

The exponential distribution arises when the randomized American option is assumed to mature at the first arrival (jump) of a Poisson process with intensity $\lambda = 1/T$; in this case the random maturity $\tilde{T}$ is exponentially distributed with expected value equal to the actual maturity $T$. Formally, the probability that the maturity lies in the interval $[t, t + dt]$ is

\[
P \left( \tilde{T} \in [t, t + dt] \right) = \lambda e^{-\lambda t} dt, \quad t \geq 0. \tag{22}
\]

This random horizon valuation problem is equivalent to an infinite horizon problem with an adjusted discount rate. Hence randomizing the maturity will lead to simpler valuation formulae.

Since the exponential distribution has a memoryless property, the early exercise boundary of this randomized American option turns out to be independent of time; this entails that the search for a boundary can be carried out over constant boundaries only.

The fair value of the randomized American put with an exponential distributed maturity is the solution of the following problem of first passage time through a constant barrier (note that the supremum is taken only over time-stationary boundaries)

\[
P_0 = \sup_{H} \mathbb{E} \left[ e^{-rt_H} \left( X - S_{t_H} \right)^+ \right] \tag{23}
\]

where $t_H$ is the first passage time through the constant barrier $H$

\[
t_H = \inf \{ \{t \in [0, T] : S_t \leq H \} \cup \{\tilde{T}\} \}. \tag{24}
\]

The expectation in equation (23) can be evaluated in closed form, and the result can be maximized over constant barriers $H$. The randomized American put value can then be written as the solution of the problem

\[
P_0 = \sup_{H} \lambda \int_{0}^{\infty} e^{-\lambda t} D_0(S, H, t)dt \tag{25}
\]

where $D_0(S, H, t)$ is the initial value of a down-and-out put with fixed maturity $t$, out barrier $H$, and rebate $X - H$. The value (25) of the randomized American put option is the Laplace-Carson transform of a fixed maturity barrier put, maximized over constant barriers $H$ ([Carr, 1998]).
The optimal (constant) exercise boundary is given by

\[ B_t = B^* = X \left( \frac{\pi_1 RrT}{\pi_2 - R\pi_1} \right)^{\frac{1}{\eta + \varepsilon}} \]  

(26)

where \( R = 1/(1 + rT), \eta = 1/2 - r/\sigma^2, \varepsilon = \sqrt{\eta^2 + 2/(R\sigma^2T)}, \pi_1 = (\varepsilon - \eta)/(2\varepsilon), \pi_2 = (\varepsilon - \eta + 1)/(2\varepsilon) \).

The assumption of an exponentially distributed maturity leads to an approximation of the put option price which entails too much errors to be used in practice. To improve this approximation, [Carr, 1998] proposes to use for the random option maturity the gamma distribution with mean \( T \) and variance \( T^2/n_a \) for a given integer \( n_a \). This distribution comes from the assumption that the random maturity occurs at the \( n_a \)-th arrival of a standard Poisson process with intensity \( \lambda = n_a/T \).

In this case, the exercise boundary of the randomized put takes the form of a staircase; the randomized American put value and the initial critical stock price \( B_0 \) can be determined using a dynamic programming algorithm.

The staircase values of the critical prices of this randomized put are determined recursively for \( m = 1, \ldots, n_a \) as follows

\[ H_m = X \left( \frac{\pi_1 RXr\Delta}{c(m) - A(m)} \right)^{\frac{1}{\eta + \varepsilon}} \]  

(27)

where \( \Delta = T/n_a \),

\[ c(m) = \sum_{l=0}^{m-1} \left( \frac{m - 1 + l}{m - 1} \right) X \left[ \pi_2^m (1 - \pi_2)^l - R^m \pi_1^m (1 - \pi)^l \right], \]  

(28)

\[ A^{(1)} = 0 \]  

and for \( m \geq 2 \)

\[ A^{(m)} = \sum_{j=2}^{m} \left( \frac{X}{H_{m-j+1}} \right)^{\eta + \varepsilon} \sum_{k=0}^{j-1} \left( 2\varepsilon \ln(H_{m-j+1}) \right)^k \sum_{l=0}^{j-k-1} \left( \frac{H_{m-j+1}}{l!} \right)^l \left( \sum_{j-1}^{j-1} \pi_j^l (1 - \pi_1)^{k+l} R^j Xr\Delta. \right. \]  

(29)

The critical price at time \( t = 0 \) of the randomized put is \( B_0 = H_{n_a} \). We can adopt this critical price as an approximation for the value of the initial critical price \( B_0 \) of a put option with (fixed) maturity \( T \).

The critical prices at the different times \( t > 0 \) can be obtained analogously, by considering randomized puts with maturity \( T-t \). A reasonable procedure to recover the boundary \( B \) suggests to discretize the time interval \([0, T]\) by choosing \( n \) equally spaced points of amplitude \( \Delta t = T/n \) and compute the boundary approximation in correspondence with the times \( 0, \Delta t, 2\Delta t, \ldots, T \).

The variance of the random maturity, \( T^2/n_a \), decreases with the number of arrivals \( n_a \) considered and tends to zero as \( n_a \) tends to infinity. Hence, the accuracy of the solution can probably be improved by increasing the number of arrivals \( n_a \) (see [Carr, 1998]), but at the expense of a greater computational cost. In order to speed up the convergence of the results to the true values, Carr suggests to use the Richardson extrapolation technique.
The $N$-point Richardson extrapolation of the initial critical price $B_0$ can be computed using the following weighted average of $N$ approximate values

$$B_0^R = \sum_{n_a=1}^{N} \frac{(-1)^{N-n_a} n_a^N}{n_a!(N-n_a)!} B_0^{(n_a)},$$

(30)

where $B_0^{(n_a)}$ denotes the initial critical stock price determined assuming $n_a$ random arrivals.

With the Richardson approximation (30), the critical stock prices approximate the true values in a computationally efficient manner.

The randomization approach has been extended also to American puts and calls on dividend paying assets; see [Carr, 1998].

6 Bunch and Johnson’s approach


They start from the observation that the critical price $B_t$ at time $t$ is the highest value of the underlying asset price at which the put value does not depend on time to maturity.

On the other hand, when it is optimal to exercise the option immediately the put value is $X - S_t$, which does not depend on how much time is left to maturity. It follows that when $S_t = B_t$ the partial derivative of the put price with respect to the time to maturity $\tau = T - t$ must be zero

$$\frac{\partial P_t}{\partial \tau} \bigg|_{S_t = B_t} = 0.$$  (31)

Let us consider first the critical price at current time $t = 0$.

The partial derivative (31) can be computed by exploiting the property that the value $P_0$ of an American put can be written as the sum of two terms: the value $p_0$ of the European put with analogous characteristics and an early exercise premium (see [Kim,1990])

$$P_0 = p_0 + \int_0^T r X e^{-rt} N(-d_2(S_0, B_t, t)) \, dt,$$  (32)

where

$$d_2(S_0, B_t, t) = \frac{\log S_0 / B_t + (r - \sigma^2/2)t}{\sigma \sqrt{t}}.$$  (33)

Let us note that the following identity holds for all $t$

$$S_t N'(d_1) = X e^{-rt} N'(d_2)$$  (34)

where $d_1$ and $d_2$ are defined as follows

$$d_1 = \frac{\log S_t / X + (r + \sigma^2/2)t}{\sigma \sqrt{t}}$$  (35)

$$d_2 = d_1 - \sigma \sqrt{t}.$$  (36)
By imposing that the partial derivative of function (32) with respect to $T$ is equal to 0 in correspondence with $S_0 = B_0$ (under the assumption that the put option price $P_t$ is continuously differentiable with respect to time) and using the identity (34), with some little algebra we obtain the following condition

$$- \int_0^T \frac{rX}{\sigma \sqrt{t}} e^{-rt} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt + \frac{\sigma}{2\sqrt{T}} B_t N'(d_1(B_0, X, T)) = 0,$$

(37)

where $\dot{B}_t = \partial B_t / \partial t$ and the generalized integral is assumed well defined. From equation (37) we obtain

$$N'(d_1(B_0, X, T)) = 2 \sqrt{T} \frac{X}{\sigma B_0} \int_0^T \frac{rX}{\sigma \sqrt{t}} e^{-rt} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt.$$

(38)

and then, remembering that

$$N'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

(39)

we find

$$\frac{1}{\sqrt{2\pi}} e^{-d_1(B_0, X, T)/2} = 2 \sqrt{T} \frac{X}{\sigma B_0} \int_0^T \frac{rX}{\sigma \sqrt{t}} e^{-rt} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt.$$

(40)

Therefore condition (37) is equivalent to the following equation

$$d_1(B_0, X, T)^2 = 2 \log \frac{\sigma^2 B_0}{2 \sqrt{2\pi T} X \int_0^T \frac{rX}{\sigma \sqrt{t}} e^{-rt} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt}. $$

(41)

By substituting for the value of $d_1(B_0, X, T)$ associated to the value $S_0 = B_0$ and taking the square root of both members of equation (41) we get

$$\left| \log \frac{B_0}{X} + \left( r + \frac{\sigma^2}{2} \right) T \right| = \sigma \sqrt{T} \left\{ 2 \log \frac{\sigma^2 B_0}{2 \sqrt{2\pi T} X \int_0^T \frac{rX}{\sigma \sqrt{t}} e^{-rt} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt} \right\}.$$

(42)

A major problem is to evaluate the integral in equation (42). Let us first observe that the integrand function can be split into the product of continuous positive functions. This means that there exists a real number $\xi \in [0, T]$ such that

$$\int_0^T \frac{e^{-rt}}{\sqrt{t}} N'(d_2(B_0, B_t, t)) \frac{\dot{B}_t}{B_t} dt = \frac{e^{-r\xi}}{\sqrt{\xi}} N'(d_2(B_0, B_0, \xi)) \int_0^T \frac{\dot{B}_t}{B_t} dt.$$

(43)

It can be noted that

$$\int_0^T \frac{\dot{B}_t}{B_t} dt = \left[ \log B_t \right]_0^T = \log B_T - \log B_0 = \log \frac{X}{B_0}.$$

(44)

Moreover, formula (42) can be simplified by introducing the following approximation for $d_2(B_0, B_0, \xi)$

$$d_2(B_0, B_0, \xi) = d_2(B_0, B_0, \xi) = \left( r - \frac{\sigma^2}{2} \right) \frac{\sqrt{\xi}}{\sigma} = \left( r - \frac{\sigma^2}{2} \right) \frac{\sqrt{\alpha T}}{\sigma},$$

(45)
where $\alpha = \xi / T \in [0, 1]$. Notice that approximation (45) corresponds to consider in the expression of $d_2$ a constant boundary function.

Using equations (39), (44) and approximation (45), the integral (43) can be approximated as follows:

$$\int_0^T \frac{e^{-rt}}{\sqrt{t}} N'(d_2(B_0, B_t, t)) \frac{B_t}{B_t} dt \simeq \frac{e^{-r \alpha T}}{\sqrt{\alpha T} \sqrt{2\pi}} e^{-(r - \sigma^2/2)(2\alpha T/(2\sigma^2))} \log \frac{X}{B_0} = \frac{e^{-(r + \sigma^2/2)^2/2}}{\sqrt{2 \pi} \sqrt{\alpha T}} \log \frac{X}{B_0}. \quad (46)$$

By using this approximation, equation (42) can be written as follows

$$\frac{B_0}{X} = e^{-(r + \sigma^2/2)T - \sigma \sqrt{T} g_1} \quad (47)$$

where

$$g_1 = \begin{cases} g & \text{if } \frac{B_0}{X} \leq e^{-(r + \sigma^2/2)T} \\ -g & \text{if } \frac{B_0}{X} > e^{-(r + \sigma^2/2)T} \end{cases} \quad (48)$$

with

$$g = g(T) = \sqrt{2 \log \frac{2 \sqrt{\gamma}}{\sigma^2} \log \frac{X}{B_0} e^{-(r + \sigma^2/2)^2/2}}. \quad (49)$$

Equations (47)-(49), which implicitly define the critical price $B_0$, are the basis of Bunch and Johnson’s procedure.

With regard to the choice of the sign of $g_1$, Bunch and Johnson observe that $g(T) = 0$ in correspondence with the time to maturity $\tau_0$ which solves the following nonlinear equation

$$\tau_0 = \frac{2 \log \frac{2 \sqrt{\gamma}}{\sigma^2} \log \frac{X}{B_0} e^{-(r + \sigma^2/2)^2/2}}{\sigma^2(1 + \gamma) [1 - \alpha(1 + \gamma)/4]}, \quad (50)$$

where

$$\gamma = \frac{2r}{\sigma^2}. \quad (51)$$

The root $\tau_0$ can be found by using a numerical algorithm. The function $g_1$ is decreasing with $T$ and is positive for $T < \tau_0$ and negative for $T > \tau_0$.

A fairly good estimate for $\tau_0$ (and a good initial value for an iterative algorithm) is

$$\tau_0 \simeq \frac{\log \frac{1 + \gamma}{\gamma}}{r + \sigma^2/2}. \quad (52)$$

Now the problem is how to compute the value of $\alpha$. Bunch and Johnson propose two different approximations for the computation of $\alpha$. In addition, they propose also a direct approximation for the value of $g$. These alternative approximations give rise to three different approximations for the critical price $B_0$. 

14
1. The first approximation for $\alpha$ is given by

$$\alpha = 1 - \frac{A}{1 + (1 + \gamma)^2 \sigma^2 \tau/4}$$

where

$$A = \frac{1}{2} \left( \frac{\gamma}{1 + \gamma} \right)^2.$$  

2. The second one is more precise, according to Bunch and Johnson, but requires the solution of a nonlinear equation as $\alpha$ is implicitly defined by the following equation

$$\alpha = \frac{4}{1 + \gamma} \left( 1 - \frac{1}{L} \log \frac{\sqrt{\alpha}}{\gamma L} \right)$$

where

$$L = \log \frac{1 + \gamma}{\gamma}.$$  

3. The third approximation is much simpler from a computational point of view and should be used only for small values of the time to maturity. This approximation arises from the following direct approximation of $g$

$$g \simeq \sqrt{\log \frac{\sigma^2}{4ert^2 T/\alpha}}.$$  

The (approximated) boundary function for $t > 0$ can be computed as we have suggested in the previous section with regard to Carr’s procedure, i.e. by discretizing the time interval $[0, T]$ and computing the boundary approximations in correspondence with the times $0, \Delta t, 2\Delta t, \ldots, T - \Delta t$.

### 7 Numerical analyses

In order to test the applicability of the algorithms studied and the goodness of the approximations they provide, we have carried out a wide numerical analysis.

In the comparisons carried out, we have used the boundary obtained with the linear interpolation (18)–(20) in the CRR binomial method (with a high number of time steps) as benchmark which best approximates the (unknown) true optimal exercise boundary. In particular, in the experiments reported we have used a 25,000 time step lattice.

More precisely, the time interval $[0, T]$ between current time and maturity has been divided into $m = 20,000$ sub-intervals of length $T/m$, but we have made the binomial algorithm start 5,000 steps before time $t = 0$ (so that the total number of steps in the lattice is $n = 25,000$) in order to have a wide range of prices already defined since the beginning of the option life (or since the time at which the option is evaluated).

We have first compared many different values for the number $N$ of points in the Richardson extrapolation used by Carr’s method. In the experimental trials carried out, the 5-point extrapolation yielded the most accurate critical prices, so all the subsequent trials (included the ones reported here) have been computed with such a choice.
Figure 3: Approximations of the optimal exercise boundary obtained with the binomial model, Carr’s and Bunch and Johnson’s approaches for an option with strike price \( X = 100 = S_0 \), maturity \( T = 1 \), \( \sigma = 0.3 \), \( r = 0.05 \). The binomial and Carr’s boundaries are almost indistinguishable.

We have randomly generated 4 000 option valuation problems with a Monte Carlo procedure: more precisely, we have let the parameter pair \((r, \sigma)\) vary in the set \([0.01, 0.2] \times [0.1, 0.5]\). The parameter space has been partitioned with a \(4 \times 4\) grid into 16 rectangular subsets and we have randomly generated 250 \((r, \sigma)\)-pairs from each subset.

For each randomly generated problem we have computed the early exercise boundary with the binomial method, Carr’s randomization method with the 5-point Richardson extrapolation and the two Bunch and Johnson’s algorithms obtained applying both the first and second approximations proposed for the computation of \( \alpha \). Actually, we have applied also the third (and simpler) approximation for \( \alpha \), but the resulting algorithm did not work well.

The early exercise boundaries have been computed in a discrete set of \( n + 1 \) (with \( n = 100 \)) equally spaced points in the interval \([0, T]\).

For each generated problem, once computed the optimal exercise boundary with the different algorithms, we have calculated a measure of the distance between the approximated boundaries obtained with Carr’s and Bunch and Johnson’s approaches and the boundary obtained with the binomial method. This distance is considered as the approximation error of the trial.

The distance between these functions has been computed as the norm of the difference between the pair of functions compared, using the discrete norm in \( L_1 \) (see e.g. [Gautschi, 1997])

\[
\text{dist} (B^{\text{bin}}, B^{\text{Carr}}) = \sum_{i=0}^{n-1} \left| B_i^{\text{bin}} - B_i^{\text{Carr}} \right|
\]
Figure 4: Approximations of the optimal exercise boundary obtained with the binomial model, Carr’s and Bunch and Johnson’s approaches for an option with strike price $X = 100 = S_0$, maturity $T = 1$, $\sigma = 0.5$, $r = 0.20$. The binomial and Carr’s boundaries are almost indistinguishable.

\[
\text{dist} (B^{\text{bin}}, B^{BJ}) = \sum_{i=0}^{n-1} \left| B^{\text{bin}}_i - B^{BJ}_i \right|, \tag{59}
\]

where $B^{\text{bin}}$, $B^{Carr}$ and $B^{BJ}$ denote the boundary obtained with the binomial, Carr’s and Bunch and Johnson’s methods.

Actually, in the tables 1-3 which summarize the main results we have reported the distance divided by the number $n$ of points in time

\[
\frac{1}{n} \text{dist} (B^{\text{bin}}, B^{Carr}) = \frac{1}{n} \sum_{i=0}^{n-1} \left| B^{\text{bin}}_i - B^{Carr}_i \right|, \tag{60}
\]

\[
\frac{1}{n} \text{dist} (B^{\text{bin}}, B^{BJ}) = \frac{1}{n} \sum_{i=0}^{n-1} \left| B^{\text{bin}}_i - B^{BJ}_i \right|, \tag{61}
\]

which corresponds to a use a weighted norm (with constant weight $1/n$). This quantity is the average point error and therefore it is easy to interpret; moreover, it allows a comparison between trials which rely on a different number of time intervals.

For each generated problem, 5 different maturities have been considered: 1, 3, 6, 9, 12 months.

First of all, we must say that the third approximation proposed by Bunch and Johnson has proved too poor to be used in practice, in all the empirical trials carried out. Moreover, except for very short maturities, it often gives a boundary which does not satisfy some of the basic properties of the optimal exercise boundary. For this reason, in the following analysis we limit our attention to the first and second approximations (denoted here by BJ1 and BJ2, respectively).
Table 1: Average distances (formulae (60) and (61)) between Carr’s, BJ1 and BJ2 boundary approximations and the binomial boundary, standard deviations of these distances and per cent number of acceptable boundaries (cases), as the maturity $T$ varies. The results refer to American put options with $S_0 = X = 100$ and are computed over all the 4000 simulations for $r$ and $\sigma$.

<table>
<thead>
<tr>
<th>$T$</th>
<th>average distance</th>
<th>st.dev.</th>
<th>cases</th>
<th>average distance</th>
<th>st.dev.</th>
<th>cases</th>
<th>average distance</th>
<th>st.dev.</th>
<th>cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/12</td>
<td>0.0254</td>
<td>0.0124</td>
<td>100 %</td>
<td>0.1950</td>
<td>0.1049</td>
<td>67 %</td>
<td>0.0972</td>
<td>0.0464</td>
<td>68 %</td>
</tr>
<tr>
<td>1/4</td>
<td>0.0221</td>
<td>0.0079</td>
<td>100 %</td>
<td>0.2106</td>
<td>0.1240</td>
<td>67 %</td>
<td>0.2153</td>
<td>0.1555</td>
<td>68 %</td>
</tr>
<tr>
<td>1/2</td>
<td>0.0216</td>
<td>0.0068</td>
<td>100 %</td>
<td>0.2955</td>
<td>0.1755</td>
<td>65 %</td>
<td>0.4192</td>
<td>0.2848</td>
<td>67 %</td>
</tr>
<tr>
<td>3/4</td>
<td>0.0218</td>
<td>0.0068</td>
<td>100 %</td>
<td>0.3827</td>
<td>0.2223</td>
<td>60 %</td>
<td>0.5921</td>
<td>0.3546</td>
<td>61 %</td>
</tr>
<tr>
<td>1</td>
<td>0.0222</td>
<td>0.0070</td>
<td>100 %</td>
<td>0.4643</td>
<td>0.2672</td>
<td>56 %</td>
<td>0.7329</td>
<td>0.4049</td>
<td>58 %</td>
</tr>
</tbody>
</table>

Another feature that is evident in all the trials carried out is that the precision of the boundary obtained with Carr’s method is much higher than the precision of the boundaries obtained with Bunch and Johnson’s approximations. A representative instance is shown in figure 3, where the different accuracy of the two approaches is evident: the boundaries obtained with the binomial and Carr’s methods are almost overlapped while BJ1 and BJ2 lie at some distance.

Moreover, while Carr’s method has proven very robust, since it always gave a good approximation of the boundary, in all the trials, we have to point out that in many cases Bunch and Johnson’s technique failed to give an acceptable approximation boundary. Actually, in many cases BJ1 and BJ2 were not able to give a sufficiently regular approximation of the boundary. An example of such an irregular behaviour is shown in figure 4.

Some additional, more detailed, results of the empirical analysis carried out are summarized in tables 1 to 3.

Table 1 reports, for the 5 maturities analyzed, the average distance between the (benchmark) binomial boundary and the approximated boundaries obtained with Carr’s method, BJ1 and BJ2, the standard deviation of these distances and the per cent number of trials which give an acceptable boundary approximation. As can be seen, the boundary obtained with Carr’s method is much closer to the binomial boundary than the boundaries obtained with BJ1 and BJ2; moreover, the accuracy of Carr’s approximation does not vary with time to maturity. On the other hand, we can observe that the number of acceptable cases for BJ1 and BJ2 procedures decreases as time to maturity increases. On the average, BJ2 results are slightly better that the ones given by BJ1 for very short maturities but they are worse for longer maturities.

Table 2 reports, for the different parameter intervals and for maturities of 1 and 12 months, the per cent number of acceptable boundaries given by BJ1 and BJ2. BJ2 is slightly more robust that BJ1. We can note that these two approximations are fairly robust for the couples $(r, \sigma)$ in the north-east corner of the grid of the parameter space (i.e., for low values of $r$ and high values of $\sigma$) while they fail almost always in the opposite corner of the grid. It is also evident from
Table 2: Per cent number of acceptable boundaries obtained with BJ1 and BJ2 methods as $\sigma$ and $r$ vary in the intervals indicated, for American put options with maturities 1 month and 1 year. The results refer to 250 simulated problems in each subinterval.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$r$</th>
<th>$[0.1, 0.2]$</th>
<th>$[0.2, 0.3]$</th>
<th>$[0.3, 0.4]$</th>
<th>$[0.4, 0.5]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T=1/12$</td>
<td>BJ1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[0.01, 0.03]$</td>
<td>70</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.03, 0.06]$</td>
<td>12</td>
<td>95</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.06, 0.10]$</td>
<td>0</td>
<td>45</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.10, 0.20]$</td>
<td>0</td>
<td>0</td>
<td>49</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>BJ2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[0.01, 0.03]$</td>
<td>74</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.03, 0.06]$</td>
<td>16</td>
<td>97</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.06, 0.10]$</td>
<td>0</td>
<td>51</td>
<td>100</td>
<td>100</td>
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</tr>
<tr>
<td>$[0.10, 0.20]$</td>
<td>0</td>
<td>2</td>
<td>55</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$T=1$</td>
<td>BJ1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[0.01, 0.03]$</td>
<td>70</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
</tr>
<tr>
<td>$[0.03, 0.06]$</td>
<td>12</td>
<td>95</td>
<td>100</td>
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</tr>
<tr>
<td>$[0.06, 0.10]$</td>
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<td>45</td>
<td>59</td>
<td>97</td>
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</tr>
<tr>
<td>$[0.10, 0.20]$</td>
<td>0</td>
<td>0</td>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>BJ2</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$[0.01, 0.03]$</td>
<td>74</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td></td>
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<tr>
<td>$[0.03, 0.06]$</td>
<td>16</td>
<td>97</td>
<td>100</td>
<td>100</td>
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<tr>
<td>$[0.06, 0.10]$</td>
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<td>8</td>
<td>14</td>
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</table>
Table 3: Average distances between Carr’s, BJ1 and BJ2 boundary approximations and the binomial boundary, as $\sigma$ and $r$ vary in the intervals indicated, for American put options with maturities 1 month and 1 year. The results refer to 250 simulated problems in each subinterval.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma$</th>
<th>[0.1, 0.2]</th>
<th>[0.2, 0.3]</th>
<th>[0.3, 0.4]</th>
<th>[0.4, 0.5]</th>
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<tr>
<td><strong>Carr</strong></td>
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<td></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>[0.01, 0.03]</td>
<td></td>
<td>0.0127</td>
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<td>0.0233</td>
<td>0.0295</td>
<td>0.0347</td>
</tr>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>[0.01, 0.03]</td>
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<td>0.0249</td>
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<td>0.0332</td>
</tr>
<tr>
<td><strong>BJ1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>T=1/12</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.3348</td>
<td>0.4518</td>
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</tr>
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<td><strong>T=1/12</strong></td>
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<td>1.1863</td>
<td>1.2629</td>
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Table 2 that the per cent number of acceptable boundaries tends to decrease as time to maturity increases.

Table 3 reports the average distances (60)-(61) for the different parameter intervals and for maturities of 1 and 12 months. It is evident that the precision of the approximation of the boundary computed with Carr’s method is practically independent of time to maturity. On the contrary, the accuracy of the boundaries obtained with Bunch and Johnson’s procedures tends to get worse as time to maturity increases.

Moreover, we have to point out that the computational results have always given approximated boundaries that are convex functions of time. Furthermore, we have checked whether the approximated boundaries were always above or below the benchmark binomial boundary. The computational results do not highlight a clear tendency with regard to this.

8 Concluding remarks

In this contribution we have analyzed three different numerical approaches which enable to approximate the optimal exercise boundary of an American put option: an improved lattice method, a randomization approach based on an American option valuation procedure proposed by Carr and an analytic approximation procedure proposed by Bunch and Johnson.

The three techniques studied are tested and compared through a wide empirical analysis carried out by considering a number of different option valuation problems generated through a simulation procedure. The comparisons have been carried out by considering the interpolated boundary obtained with a binomial method (with a high number of time steps) as benchmark which best approximates the true optimal exercise boundary.

From the experiments performed it comes out that Carr’s method is efficient, robust and precise while Bunch and Johnson’s methods are both less robust and less precise, above all on the longest maturities. On the other hand, the computational times required by Carr’s and Bunch and Johnson’s procedures are negligible, compared with the computational time needed by a lattice procedure. With regard to this, the problem is that a lattice method needs a large number of steps in order to provide a sufficiently smooth exercise boundary.

The computation of the optimal boundary allows to effectively use the optimal stopping rule for the early exercise of American options. This optimal stopping rule can be applied within a forward Monte Carlo simulation procedure, in order to determine not only the option value but also the average optimal stopping time and the probability that the option expires without having been exercised early.

References


